seaborn

CEE 260/MIE 273: Probability and Statistics in Civil Engineering

Lecture M7a: Correlation and Least Squares Estimation

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Outline

Today's objectives

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- Learn how to compute and interpret the correlation coefficient
- Understand and apply linear regression
- Analyze regression fitness metrics (in particular, R^2)

Historical note: regression analysis

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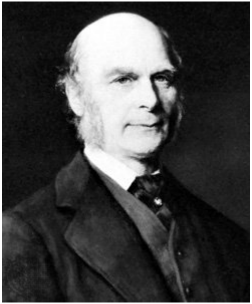

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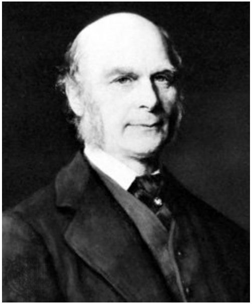

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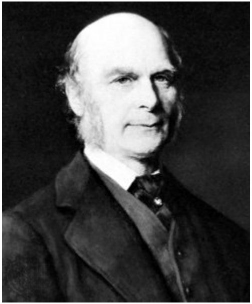

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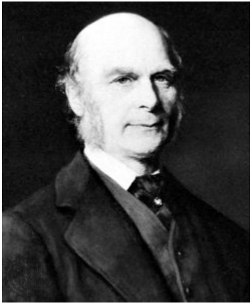

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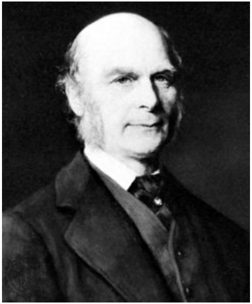

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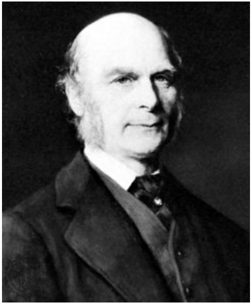

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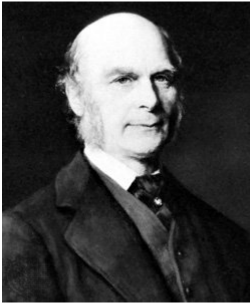

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  - E.g. the height of the son of an taller-than-average father was greater than average but not by as much as his father's
  - And the height of the son a shorter-than-average father was lower but not by as much as his father's

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Examples of variables with nondeterministic relationships

- $x =$ age of a child; $y =$ size of child's vocabulary
- $x =$ size of an engine; $y =$ fuel efficiency for a car equipped with engine
- $x =$ applied tensile force; y deformation of a metal strip

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In this module, we will cover the following key topics:

- Correlation and variance analyses
- Simple Linear Regression and Least Squares Estimation
- Inference for Linear Regression

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This can also be rewritten as:

$$\text{Cov}(X, Y) = \begin{cases} \sum_x \sum_y (x - \mu_x)(y - \mu_y) f(x, y) & (X, Y) \text{ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x - \mu_x)(y - \mu_x) f(x, y) dxdy & (X, Y) \text{ continuous} \end{cases} \quad (3)$$

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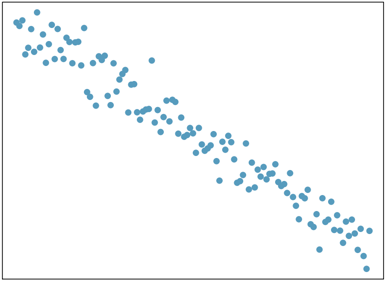
Thus, the correlation coefficient ranges from -1 (perfectly linear negative relationship) to $+1$ (perfectly linear positive relationship).



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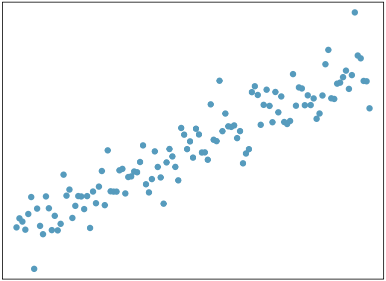

(b)

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where

$$S_{xy} = \sum (x_i - \overline{x})(y_i - \overline{y}) \tag{6}$$

$$S_{xx} = \sum (x_i - \overline{x})^2 \tag{7}$$

$$S_{yy} = \sum (y_i - \overline{y})^2 \tag{8}$$

Properties of the sample correlation coefficient $\hat{\rho}$

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1. Value does not depend on which of the two variables is labeled x or y
2. Independent of the units in which x and y are measured
3. $-1 \leq \hat{\rho} \leq 1$
4. $\hat{\rho} = 1$ if and only if all data pairs lie on a straight line with positive slope and $\hat{\rho} = -1$ iff[1] all pairs lie on a straight line with negative slope

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&= \frac{\sum x_i y_i - n\bar{x}\,\bar{y}}{\sqrt{\sum x_i^2 - n\bar{x}^2}\sqrt{\sum y_i^2 - n\bar{y}^2}}
\end{aligned}
$$

Computing the correlation coefficient from summary data

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\sum x_i y_i &= 20.0397
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\sum y_i^2 &= 2253.56
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Find the sample correlation coefficient.

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Example 1: Pollutant concentrations (cont.)

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&= \frac{20.0397 - \frac{1}{16}(1.656)(170.6)}{\sqrt{0.196912 - \frac{1}{16}(1.656)^2}\sqrt{20.0397 - \frac{1}{16}(170.6)^2}}
\end{aligned}$$

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&= \frac{20.0397 - \frac{1}{16}(1.656)(170.6)}{\sqrt{0.196912 - \frac{1}{16}(1.656)^2}\sqrt{20.0397 - \frac{1}{16}(170.6)^2}} \\
&= \boxed{0.716}
\end{aligned}
$$

Simple linear regression model

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$$y = \beta_0 + \beta_1 x + \epsilon \tag{13}$$

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β_0 (intercept) and β_1 (slope) are the **regression coefficients**

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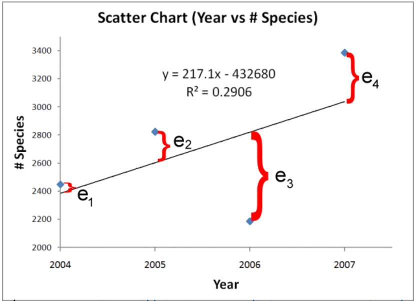

Image source: https://sigmazone.com/labrea_scatter_plots/

The observed errors in model predictions are known as **residuals**.

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Example: Age and vocabulary size of children

Let:

$$x = \text{age of a child}$$
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Then $\mu_{Y\cdot 5}$ is the average vocabulary size for all 5-year-old children in the population.
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- The amount of variability in Y values is the same for each value of x (**homogeneity of variance**)

Analyzing a regression equation

Example 2: Flow rate

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The flow rate y (m^3/min) in a device used for air quality measurement depends on the pressure drop x (in. of water) across the device's filter. Suppose that for x values between 5 and 20, the variables are related by the regression model:

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(c) What is the expected flow rate for a pressure drop of 10 in?

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(d) For a pressure drop of 10 in., what is the probability that the observed flow rate will exceed 0.835?

Analyzing a regression equation

Example 2: Flow rate (cont.)

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Example 2: Flow rate (cont.)

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(a) The **expected change** in flow rate associated with a 1-inch increase in pressure drop is the slope:

Analyzing a regression equation

Example 2: Flow rate (cont.)

From the model equation: $\beta_0 = -0.12$ and $\beta_1 = 0.095$

(a) The **expected change** in flow rate associated with a 1-inch increase in pressure drop is the slope: 0.095 m^3/min.

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$$0.095(-5) = -0.475\ \text{m}^3/\text{min}$$

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Example 2: Flow rate (cont.)

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(d) For $x = 10$, Y has a mean value $\mu_{Y \cdot 10} = \mathbb{E}(Y|X = 10) = 0.83$ (from part (c)).

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Example 2: Flow rate (cont.)

(d) For $x = 10$, Y has a mean value $\mu_{Y \cdot 10} = \mathbb{E}(Y|X = 10) = 0.83$ (from part (c)).
Thus:

$$P(Y > 0.835|X = 10) \quad = \quad 1 - P(Y \leq 0.835|X = 10)$$

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Analyzing a regression equation

Example 2: Flow rate (cont.)

(d) For $x = 10$, Y has a mean value $\mu_{Y \cdot 10} = \mathbb{E}(Y|X = 10) = 0.83$ (from part (c)).
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Least squares regression in MATLAB

Example 3: Relationship between population and number of accidents

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Using the `accidents` dataset in MATLAB, perform a least-squares regression of accidents in a state *on* the population of the state[a]:

```
load accidents
x = hwydata(:,14); (Population of state)
y = hwydata(:,4); (Accidents per state)
```

(a) What are the slope and intercept estimates, $\hat{\beta}_0$ and $\hat{\beta}_1$?

(b) How can you evaluate the strength of the relationship?

[a] See the file `ex3_l19_least_squares_regression.m`

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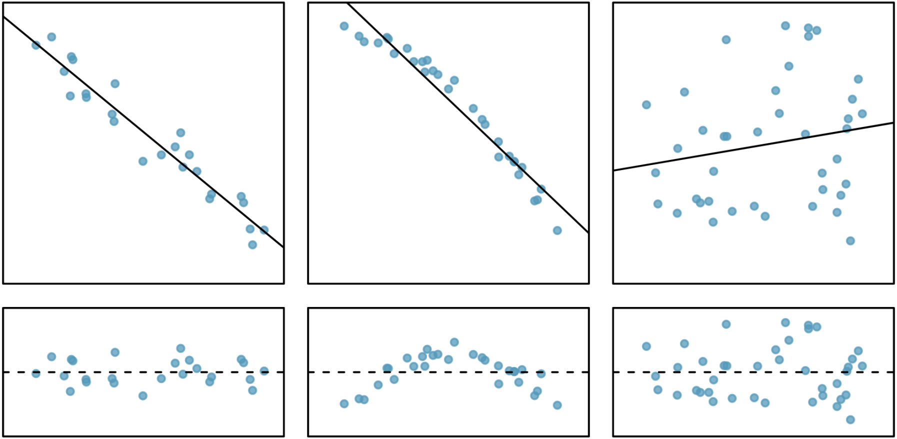

What patterns do you observe?

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The degrees of freedom $df = n - 2$ because 2 parameters must first be estimated before computing $\hat{\sigma}^2$: β_0 and β_1

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- The ratio SSE/SST is the proportion of total variation unexplained by the simple linear regression model

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$$R^2 = 1 - \frac{SSE}{SST} \tag{22}$$

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- $\hat{\rho}^2$ approximates R^2 for large n

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\textbf{Coefficient of determination} \quad R^2 &= 1 - \frac{SSE}{SST} = \frac{SSR}{SST} && (30)
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Note

The variance estimate $\hat{\sigma}^2$ is also defined as the *conditional variance*, $\mathbb{V}(Y|X = x)$